

ON THE COMPLETENESS OF QUANTUM MECHANICS AND THE INTERPRETATION OF THE STATE VECTOR*

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Recently, it has been argued that quantum mechanics is complete, and that quantum states vectors are necessarily in one-to-one correspondence with the elements of reality, under the assumptions that quantum theory is correct and that measurement settings can be freely chosen. In this work, we argue that the adopted form of the free choice assumption is stronger than needed. In our perspective, there are hidden assumptions underlying these results, which limit their range of validity. We support our argument by a model for the bipartite two-level system, reproducing quantum mechanics, in which the free will assumption is respected, and different quantum states can be connected to the same state of reality.

I. INTRODUCTION

Quantum mechanics is one of the most successful theories ever developed. It has been elaborated to account for the phenomena of the microscopic world and it has been experimentally verified to an impressive degree of accuracy. It has required a radical change in the scientific conception of nature particularly in connection with its indeterminism and nonlocality. However, from a conceptual point of view, the theory is not immune of drawbacks. The crucial point stays in the fact that it appears more as a set of operational prescriptions to predict outcomes of prospective measurement procedures than a coherent description of

*This research is supported by the ARO MURI grant W911NF-11-1-0268.

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reality. Its range of validity is not clear, it relies on two different kinds of evolution depending on the rather vague notion of measurement. As a consequence of these problematic aspects it is not clear what is the conceptual status of the quantum state vector ψ , the mathematical entity associated to the state of any physical system. After the famous incompleteness argument by Einstein, Podolski and Rosen [1], the question of whether ψ represents a state of reality or rather a state of knowledge, as suggested by its updating following a measurement procedure, remains unanswered.

In order to address some of these questions, several *ontological models* of quantum mechanics have been introduced. These are theories predictively equivalent to quantum mechanics, but providing a possibly richer description of microscopic reality in terms of the so-called *ontic state*, i.e., the most accurate specification of the physical state of the system [2]. In these theories, the state vector ψ might embody only partial information on the ontic state. In particular, it is associated to a distribution $\rho_\psi(\lambda)$ on the space of the ontic variables, with $\rho_\psi(\lambda) \geq 0$ and

$$\int \rho_\psi(\lambda) d\lambda = 1 \quad \text{for all } \psi. \quad (1)$$

The ontic state λ contains all the information about the elements of reality of the underlying theory, and, accordingly, it provides the complete description of the state of the system [25]. In principle, λ can be not fully accessible. Ignorance of its precise value is deemed to be the origin of various of the aforementioned problematic aspects of quantum mechanics.

In the past literature, ontological models were usually referred to as *hidden variables theories*, meaning that additional variables, absent in the standard quantum formalism (and for this reason named “hidden”), were expected to supplement the information given by ψ , providing in this way a complete description of the physical situation. From the beginning, this program had supporters as well as opponents. J. von Neumann proved that deterministic hidden variables models were unviable [3], but later D. Bohm provided an explicit, fully deterministic, hidden variable model predictively equivalent to quantum mechanics [4, 5]. Later J.S. Bell appropriately stressed that von Neumann’s impossibility theorem was circular, since it was based on logically unnecessarily strong assumptions [6]. He also proved that any theory predictively equivalent to quantum mechanics must violate some inequalities, named *Bell’s inequalities* after him, which are satisfied by any local theory [7].

Recently, a new argument supporting the completeness of quantum mechanics has been presented in [8], based only on two assumptions: that quantum mechanics is correct (*QM*),

and that a precise mathematical condition (FR) - which the authors believe to express the request that measurement settings can be freely chosen - be satisfied. According to [8], any completion of quantum mechanics which consistently reproduces its outcomes, necessarily comes at the expenses of the free will of the observers in choosing the measurement settings. Here, we argue that, similarly to the case of von Neumann argument against hidden variables theories, also this result is inconclusive due to the inappropriate formalization of the assumption (FW) of free will, which we will make precise in what follows. Specifically, we provide evidence that the physically meaningful request of FW is not correctly expressed in mathematical terms in [8], since the constraint FR , which stays at the basis of the authors' argument, is unnecessarily stronger than FW . For related comments see [9–11].

Our criticism has relevant consequences concerning the meaning of the quantum state vector ψ in ontological models. First of all, let us recall that the distributions $\rho_\psi(\lambda)$ might have overlapping supports for different ψ , a property which characterizes the so-called *ψ -epistemic models*. Alternatively, when the supports are always disjoint we are dealing with *ψ -ontic models* [2]. According to this classification the state vector ψ acquires a strongly different conceptual status. In fact, in a ψ -ontic theory different state vectors necessarily correspond to different ontic states, whereas in a ψ -epistemic theory they could correspond to the same ontic state. As a consequence, one can associate to ψ well defined elements of reality only in the first case.

As it should be obvious, the distinction between ψ -epistemic and ψ -ontic models implies completely different interpretations of the quantum state vector. For this reason, these classes of models have recently been the subject of several investigations. A first argument arguing that ψ -epistemic models necessarily contrast with quantum mechanics was developed in [12], but its conclusions are severely limited by the assumption that factorized quantum states correspond to factorized states of the underlying theory. In fact, an explicit ψ -epistemic model has been built by relaxing this assumption [13]. Another no-go theorem for ψ -epistemic models has been developed in [14], based on the completeness result presented in [8]. Our arguments against the assumption FR make also this argument inconclusive in determining the ontological status of the quantum state vector, since a weaker and more significant condition of free will is still compatible with ψ -epistemic models. We substantiate our argument by presenting an explicit ψ -epistemic model for a pair of two-level systems, in which the experimenters can freely choose their local settings, so that FW holds, even

though condition FR is violated.

Before proceeding, a relevant remark is at order. Here, and anywhere in this work, when using the term *reality* we do not mean *what the world is*, but rather *what the theory is about*, exactly in the spirit of Bell's beables [15]:

I use the term beable rather than some more committed term like being or beer to recall the essentially tentative nature of any physical theory.

We further assume that a complete description can be obtained by confining our attention to a specific set of physical systems, the rest of the universe not contributing at all, as appropriately required by Shimony, Horne and Clauser [16]:

Unless we proceed under the assumption that hidden conspiracies [...] do not occur, we have abandoned in advance the whole enterprise of discovering the laws of nature by experimentation.

Accordingly, we do not consider superdeterministic models in our analysis.

In Section II we review the completeness argument of [8], and its implications on the interpretation of the state vector [14], by slightly modifying the original notations for sake of clarity. We only describe the steps which are relevant to our analysis, and refer the interested reader to the original papers for further details. In Section III we criticize the assumption that the condition FR introduced in [8] correctly embodies the request FW of free will. Actually, we shall prove that FR implies more than the free will of the experimenters, and this will make clear how our analysis limits the results of [8, 14]. In Section IV we substantiate our arguments by providing an explicit model in which the quantum state vector is not in one-to-one correspondence with the elements of reality. Finally, in Section V, we criticize a more recent argument supporting the completeness of quantum mechanics, and we present our final remarks.

II. THE COMPLETENESS ARGUMENT OF COLBECK AND RENNER

We consider the standard EPR-like scenario: two space-like separated observers perform local measurements on the two parties of an entangled state ψ . The measurement settings on the two sides are given by vectors A and B , and the measurement outcomes by X

and Y , whose values are in $\{-1, 1\}$. We assume that additional information on the ontic state λ can be accessed through a measurement procedure with setting C and output Z . All these quantities are assumed to be space-time random variables, whose values can be associated to well-defined space-time points (t, r_1, r_2, r_3) , making precise the statement that two measurement procedures and the associated outcomes are space-like with respect to one another. All these quantities are described by joint conditional probabilities of the form $P_{XYZ|ABC}$, and similar ones. Generally, these probabilities are conditioned by ψ , but usually this dependence will be left implicit for notational convenience.

The argument presented in [8] relies essentially on two assumptions, which we now review. The first one, denoted by FR , is that the observers can independently choose the measurements they perform. As explicitly stated in [8]:

Assumption FR is that the input, A , of a measurement process can be chosen such that it is uncorrelated with certain other space-time random variables, namely all those whose coordinates lie outside the future light-cone of the coordinates of A .

This assumption holds true for B and C as well, and it is mathematically expressed (and this is actually what the author's mean by the request FR) through the following requirements on the conditional probabilities:

$$P_{A|BCYZ} = P_A, \quad P_{B|ACXZ} = P_B, \quad P_{C|ABXY} = P_C. \quad (2)$$

The second assumption, denoted by QM , is that quantum mechanics is correct [8]:

Measurement outcomes obey quantum statistics and [...] all processes within quantum theory can be considered as unitary evolutions, if one takes into account the environment. [Moreover] the second part of the assumption need only hold for microscopic processes on short timescales.

Now, by relying only on these two assumptions, the authors of [8] prove that any knowledge on the ontic state besides the one given by ψ is useless, since it cannot modify the outcome distributions predicted by quantum mechanics: $P_{XY|ABCZ} = P_{XY|AB}$, $P_{X|ABCZ} = P_{X|A}$ and $P_{Y|ABCZ} = P_{Y|B}$. They conclude that [8]

Any attempt to better explain the outcomes of quantum measurements is destined to fail. Not only is the universe not deterministic, but quantum theory provides the ultimate bound on how unpredictable it is. [...] under the assumption that measurement settings can be chosen freely, quantum theory really is complete.

The technical details of their proof are not relevant here, since our criticism focuses on the adoption of the assumption FR to formalize the request FW of the possibility of freely choosing the apparatus settings. We point out that all the three constraints in (2) are needed to prove this result and that these constraints imply the non-signalling conditions:

$$P_{YZ|ABC} = P_{YZ|BC}, \quad P_{XZ|ABC} = P_{XZ|AC}, \quad P_{XY|ABC} = P_{XY|AB}, \quad (3)$$

so that no experimenter can extract, from his outcomes, any information about the measurement settings which have been chosen by the other experimenters. This fact is physically relevant since it avoids superluminal communication between them.

By adapting their argument, in [14] the same authors provide a no-go theorem for ψ -epistemic models, ultimately based on the same assumptions FR and QM . They show that $P_{X|\lambda A} = P_{X|\psi A}$, and then conclude that there should be a one-to-one correspondence between λ and ψ . As before, the technical machinery leading to this conclusion is irrelevant here, but we stress once more that also this result is a direct consequence only of the two assumptions: FR and QM . In the following, we assume that QM is fulfilled, and we criticize the identification of FR with the assumption FW of the free choice of measurement settings, arguing that the latter can be satisfied even though FR is violated.

III. CRITICISM OF THE FR ASSUMPTION AND ITS CONSEQUENCES

As noticed by the authors of [8], their result has meaningful consequences when applied to existing hidden variables models, notably, to Bohmian mechanics. Since this is a deterministic completion of quantum mechanics, following the analysis of J.P. Jarrett, its non-locality necessarily arises from a violation of the condition of parameter independence (PI), which is expressed as

$$P_{X|AB\lambda} = P_{X|A\lambda}, \quad P_{Y|AB\lambda} = P_{Y|B\lambda}. \quad (4)$$

In the context of [8], PI reduces to $P_{X|ABZ} = P_{X|AZ}$ and $P_{Y|ABZ} = P_{Y|BZ}$, since it is assumed that the information represented by λ is fully accessible (and obtained by Z). This is by

itself a constraint which limits the applicability of the results presented in [8], since it ignores the important distinction between controllable and uncontrollable hidden variables, which has been strongly emphasized by A. Shimony [17]. In fact, it is generally assumed that the hidden variables are not completely accessible; that is, there can be some physical principles which limit their knowledge and their manipulation, ensuring, first of all, the impossibility of superluminal signalling. Maintaining the authors' perspective on the accessibility of λ , we observe that the non-signalling conditions (3) automatically imply PI , and therefore the assumption FR trivially forbids all the deterministic completions of quantum mechanics. In [8], this fact is interpreted by claiming that these models violate the assumption of free choice. Nonetheless, as we have already mentioned and as we shall shortly prove, the FR assumption cannot be considered as the appropriate formalization of the free will assumption FW , since it embodies more than this assumption. Therefore, the violation of FR cannot be automatically related to lack of free will.

To start with, we introduce the condition FW , which, in our opinion, provides a more suitable expression for the free choice assumption. We observe that, when dealing with correlation experiments of this type, one is merely interested in the freedom of choice of the measurement settings A and B , located at the two wings of the experimental set-up. This condition, when expressed in terms of conditional probabilities, reads

$$P_{A|B\lambda} = P_A, \quad P_{B|A\lambda} = P_B, \quad (5)$$

and it fully accounts for the fact that the two experimenters can freely and independently choose which observable to measure, since it implies the factorization $P_{AB\lambda} = P_AP_BP_\lambda$. Notice that the assumption FW does not involve a third party, and its measurement setting C . Actually, the authors of [8] remark that the additional information supplementing ψ

[...] must be static, that is, its behavior cannot depend on where or when it is observed [...] so, we can consider the case where its observation is also space-like separated from the measurements specified by A and B .

This statement intends to stress that FR is the appropriate way to express the free will request, since it is consistent with the condition $P_{C|ABXY} = P_C$. Nonetheless, for future reference, we take it as an independent assumption, denoted as ST , and expressed through the condition

$$P_{CZ|ABXY} = P_{CZ}. \quad (6)$$

Notice that in [8] ST does not appear as an assumption, nor the mathematical expression (6) appears somewhere: it is introduced by us in our criticism of the assumption FR .

There is another striking difference between the conditions FR and FW : the former involves the random variables X and Y , the latter does not. We believe that it is important to avoid this dependence, since the extra variables X and Y could bring spurious correlations, completely independent of the free will assumption. To clarify the idea, let us assume that we are interested in the free choice of A , and let us raise the following question: if $P_{A|BCYZ} \neq P_A$, can we conclude that A cannot be freely chosen? We do not think that this is the case. For instance, we might suppose that the two Stern-Gerlach apparatuses located at the opposite wings of the experiment could superluminally communicate, at a suitable finite speed. In the rest frame of the two experimenters, the communication happens after the free choice of A and B , and before the generation of the outputs X and Y . In this way, correlation between A and Y might be produced. Even though it is highly implausible that a physical process as the one just mentioned has any physical meaning, its consideration has nothing to do with the fact that A and B can be freely and independently chosen. In other words, we believe that a good mathematical formulation of the free measurements choice should not automatically reject situations where free will and superluminal communication coexist. Unfortunately, this is not the case when the FR assumption is considered, since it implies the non-signalling constraints (3), as stated in Section II.

In accordance with the previous analysis, we find convenient to write non-signalling conditions (which are independent from the additional information on the ontic state, and from the setting of its measuring device) as follows:

$$P_{X|AB} = P_{X|A}, \quad P_{Y|AB} = P_{Y|B}. \quad (7)$$

They are weaker than relations (3), still they fully express the impossibility of superluminal communication in the standard EPR scenario. In the following, we refer to Eq. (7) as the NS assumption. As proven in [11], the relation between FR , FW , ST and NS is given by

$$FW \wedge NS \wedge ST \Rightarrow FR, \quad (8)$$

and, if the ontic state λ is fully accessible, also the inverse implication holds,

$$FR \Rightarrow FW \wedge NS \wedge ST. \quad (9)$$

Therefore, in the case considered in [8], the FR assumption is equivalent to the logical conjunction of our assumptions FW of free will, staticity ST of the ontic state, and impossibility of superluminal communication, NS . This result proves our statement that FR is more than the free choice assumption, and shows that, even if the ontic state is (at least partially) inaccessible, negation of FR does not necessarily imply absence of free will. It might depend on a violation of free will, on the fact that the additional information on λ is not static, or on a violation of the impossibility of superluminal communication. Of those, the second condition appears the easiest to digest: why the extra information on λ , which is complementing the information provided by the state vector ψ , should be static, when ψ itself, and the measurement procedure involved in its preparation, are not space-like separated with respect to A and B ?

Therefore, the completeness argument presented in [8] has not the claimed generality, although it is formally correct. When considering Bohmian mechanics, the statement that measurement settings cannot be freely chosen is not justified. If we assume that λ is fully accessible (that is, that all positions are known), it is a well known fact that the theory allows superluminal communication. Otherwise, it is likely that the additional information on the ontic state is not static (in fact, it should be a partial information on the positions, which are distributed according to ψ , which in turn is certainly a non-static quantity). In both cases, violation of FR does not require lack of free will.

The no-go theorem for ψ -epistemic theories derived in [14] is based on the assumptions QM and FR . Therefore, the same conclusions apply to this case as well, and this result is not general at all. Both the questions of whether quantum mechanics is a complete theory, and the interpretation of the quantum state as a state of knowledge or an element of reality, remain open. In the following section we present a ψ -epistemic model for a pair of two-level systems, fully consistent with quantum mechanics, in which the measurement settings A and B can be freely chosen.

IV. A ψ -EPISTEMIC MODEL FOR THE BIPARTITE SYSTEM

The model is a trivial application of the results of [13], where, however, it has not been realized that the family of ψ -epistemic models therein presented are valid beyond the single system case. Following our reasoning in the previous section, we can claim that superluminal

influences of measurement choices upon ontic variables, in the multipartite case, are not necessarily expected. Despite in [13] a system of arbitrary dimension is considered, here we limit our attention to a pair of two-level systems for sake of simplicity.

The model is inspired by a generalization, to the case of two subsystems, of a Bell model for a single two-level system. We start by describing this generalization. If we denote by \mathcal{H} the Hilbert space of the single system, an arbitrary quantum state vector is given by $\varphi \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The hidden variable is a real parameter $\tau \in T = [0, 1]$, and the ontic state is given by the ordered pair $\lambda = (\varphi, \tau) \in \mathcal{H} \times T$. Its distribution corresponding to the quantum state ψ is given by

$$\rho_\psi(\lambda) = \rho_\psi(\varphi, \tau) = \delta(\varphi - \psi), \quad (10)$$

that is, τ is uniformly distributed over T for all states ψ . We consider now the local observables $\hat{A} = A \cdot \sigma_A$ and $\hat{B} = B \cdot \sigma_B$, where the settings A and B are unit real vectors in \mathbb{R}^3 and σ_A, σ_B are the vectors of Pauli matrices acting on \mathcal{H}_A and \mathcal{H}_B respectively. We denote the common eigenvectors of the commuting observables \hat{A}, \hat{B} and $\hat{A} \otimes \hat{B}$ as ϕ_j , and we order them according to

$$|\langle \phi_j | 0 \rangle|^2 \geq |\langle \phi_{j+1} | 0 \rangle|^2 \quad \text{with } j = 0, \dots, 3, \quad (11)$$

where $|0\rangle \in \mathcal{H}$ is an arbitrary reference state. The states ϕ_j are factorized, and they are associated with well defined outcomes (X, Y) for the observables \hat{A} and \hat{B} , which are in the set $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$. The outcome of $\hat{A} \otimes \hat{B}$ is given by the product XY . The specific correspondence between j and the pairs of possible outcomes for X and Y depends on the local observables taken into account, in the way we describe now. In the ontic state $\lambda = (\varphi, \tau)$ the value of the projectors $\hat{\Phi}_j = |\phi_j\rangle\langle\phi_j|$ are defined as

$$\Phi_j(\lambda) = 1, \quad \text{if } \sum_{k=0}^{j-1} |\langle \phi_k | \varphi \rangle|^2 \leq \tau < \sum_{k=0}^j |\langle \phi_k | \varphi \rangle|^2, \quad (12)$$

and $\Phi_j(\lambda) = 0$ otherwise; when $j = 0$, the lower bound for τ is defined to be 0 (in this case the former expression is meaningless). Therefore, with any λ there is associated a value of j . Since the spectral decomposition of the local operators \hat{A} and \hat{B} (as well as that of $\hat{A} \otimes \hat{B}$) can be written through the projectors $\hat{\Phi}_j$, the corresponding outcomes $X = X(\lambda)$ and $Y = Y(\lambda)$ (and, of course, their product) are unambiguously determined. They are

distributed according to the quantum mechanical rules [13], since

$$\int d\lambda \rho_\psi(\lambda) \Phi_j(\lambda) = \int d\tau \Phi_j(\psi, \tau) = |\langle \phi_j | \psi \rangle|^2. \quad (13)$$

As the original Bell model for a single two-level system, this generalized model is ψ -ontic. In fact, according to (10), the distributions corresponding to different state vectors never overlap. However, this property is not needed at all in order to produce a model predictively equivalent to quantum mechanics: it is sufficient that the measure of the support of the function $\Phi_j(\lambda)$ over $\{(\varphi, \tau), \tau \in T\}$ equals the quantum probability $|\langle \phi_j | \psi \rangle|^2$. Following [13], by suitably redistributing this support, it is possible to turn the model into a ψ -epistemic one. After noticing that $|\langle \phi_0 | 0 \rangle|^2 \geq 1/4$, the authors introduce

$$z(\varphi) = \inf_{\phi: |\langle \phi | 0 \rangle|^2 \geq 1/4} |\langle \phi | \varphi \rangle|^2, \quad (14)$$

which is used to define a subset of the ontic space,

$$\mathcal{E}_0 = \{(\varphi, \tau) : |\langle \varphi | 0 \rangle|^2 > 3/4 \text{ and } 0 \leq \tau < z(\varphi)\}. \quad (15)$$

This set has the property that all $\lambda \in \mathcal{E}_0$ are in the support of $\Phi_0 = |\phi_0\rangle\langle\phi_0|$. Therefore, these ontic states have all the same outcome, which is the one associated with ϕ_0 . If we redistribute over the whole of \mathcal{E}_0 the probabilities which the distribution $\rho_\psi(\lambda)$ in (10) assigns to ontic states within \mathcal{E}_0 itself, we achieve the desired target. For example, by denoting by $\rho_{\mathcal{E}_0}(\lambda)$ the uniform distribution over \mathcal{E}_0 , and by $\Theta(x)$ the unit-step function, such that $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$, it is possible to define

$$\tilde{\rho}_\psi(\lambda) = \tilde{\rho}_\psi(\varphi, \tau) = \begin{cases} \delta(\varphi - \psi), & \text{for } |\langle \psi | 0 \rangle|^2 \leq 3/4 \\ \delta(\varphi - \psi)\Theta(x - z(\psi)) + z(\psi)\rho_{\mathcal{E}_0}(\lambda), & \text{for } |\langle \psi | 0 \rangle|^2 > 3/4 \end{cases} \quad (16)$$

which, by construction, satisfies

$$\int d\lambda \tilde{\rho}_\psi(\lambda) \Phi_j(\lambda) = |\langle \phi_j | \psi \rangle|^2, \quad (17)$$

but in general produces overlapping supports for distinct state vectors. Therefore, if we replace $\rho_\psi(\lambda)$ by $\tilde{\rho}_\psi(\lambda)$, we obtain a ψ -epistemic model for a pair of two-level systems. There is not reason why the conditions *FW* and *NS* introduced in the previous section should be violated, therefore we conclude that it is possible to build ψ -epistemic models which do not ask for violations of the free will assumption, or superluminal communication.

In this section we have adopted a weak notion of ψ -epistemic model: there must be at least one pair of quantum state vectors whose corresponding distributions have overlapping supports. A stronger definition requires that all non-orthogonal quantum states have overlapping distributions. While a detailed treatment of this case is out of the scopes of this paper, we point out that our remarks on the expression of the free will assumption holds true also in this case.

V. FINAL REMARKS AND CONCLUSIONS

Recently, the authors of [8, 14] have provided a different argument leading to the same conclusion of completeness of quantum mechanics [18]. This work is presented as an expanded and more pedagogical version of [8, 14], nonetheless it appears as a distinct derivation. For instance, the result that quantum mechanics is maximally informative (that is, complete), asks for different resources in the cases of maximally and non-maximally entangled states, unlike [8, 14]. Here we are not interested in a general analysis of this work, but in investigating the assumption of free choice which appears in it, which is the most significative requirement to draw the conclusions expressed in the paper. Following the authors [18],

[...] a parameter of the theory, say A , is considered free if it is possible to choose A such that it is uncorrelated with all other values (described by the theory) except those that lie in the causal future of A .

They consistently express this idea through a factorization of the probabilities $P_{A\Gamma_A} = P_A P_{\Gamma_A}$, where Γ_A is the set of all random variables which are not in the causal future of A . This condition must hold also for B , that is $P_{B\Gamma_B} = P_B P_{\Gamma_B}$, where Γ_B is the set of all random variables which are not in the causal future of B .

Differently from the former assumption FR , this condition does not involve the random variables C and Z , and therefore it is free from the former assumption of static variables, ST . Nonetheless, as FR did, also this condition automatically satisfies NS , and therefore it cannot coexist, neither in principle, with a theory allowing superluminal communication. This can be derived by expressing P_{ABX} in different ways:

$$P_{ABX} = P_{X|AB} P_{AB} = P_{X|AB} P_A P_B, \quad (18)$$

where, in the second step, we have considered that $B \in \Gamma_A$; but also

$$P_{ABX} = P_{AX}P_B = P_{X|A}P_AP_B, \quad (19)$$

where the first step follows from $A \in \Gamma_B$ and $X \in \Gamma_B$. By comparing (18) and (19), we derive the non-signalling condition $P_{X|AB} = P_{X|A}$; a similar argument proves that $P_{Y|AB} = P_{Y|B}$, and finally NS is derived.

Therefore, according to our former discussion, we believe that this is not a legitimate expression of the free will, and also the results of this work are not as general as stated. Notice that, to motivate their mathematical expression of the free will, the authors of [18] mention how J.S. Bell affirmed characterized the concept of free variables [19]:

for me this means that the values of such variables have implications only in their future light cones.

However, “having no implications” is not the same as “being uncorrelated”. The fact that I decide to go out with my umbrella is correlated with the weather forecasts, but it has not implications on them. Therefore, in our opinion, the expression of the free will adopted in [18] does not express the Bell’s point of view, and is not adequate.

For maximally entangled states, there is independent evidence that no theory consistent with quantum mechanics can produce probabilities differing from those predicted by the quantum theory [20–23]. Nonetheless, this result is in general false for non-maximally entangled states [10]. Notice that, in [18], the predictive equivalence between the quantum theory and the underlying model, in the case of non-maximally entangled states, is obtained by reducing this case to that of maximally entangled states. This step relies on the extra resource of *embezzling states*, a special set of states whose precise characterization requires an infinite-dimensional Hilbert space [24]. By using these states, the two experimenters can transform an arbitrary state of some Hilbert space into a maximally entangled state of the same Hilbert space, via local operations. We believe that the use of this resource is not legitimate in this context, since it does not allow a self-contained description of any finite-dimensional Hilbert space.

We conclude that the issue of completeness of quantum mechanics, and the problem of the interpretation of the quantum state vector, are still open.

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- [25] To clarify the problem of attributing elements of physical reality to a physical system it is useful to make reference, once more, to the EPR paper. In the case in which ψ is an eigenstate of an observable Ω pertaining to the eigenvalue ω_k , we know that the standard quantum theory predicts that a measurement of Ω gives with certainty the outcome ω_k , and, on the basis of this fact, EPR claim that the system possesses $\Omega = \omega_k$ as an element of physical reality. In such a case, with reference to the ontic state λ , we must impose that the value $\Omega(\lambda)$ of the observable Ω must be defined and must coincide with ω_k . Obviously, it may very well happen (and to which extent this actually occurs depends on the ontic theory one has in mind), that for the given ψ and λ , the knowledge of λ allows the specification of further elements of reality besides those based on the EPR criterion. Typically, ψ might be an eigenstate of the z spin component of a particle, but knowledge of λ might allow one to associate elements of physical reality also to the x and/or y spin components.